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INTEGRAL OPERATOR METHODS IN THE THEORY OF WAVE PROPAGATION AND--ETC(U)
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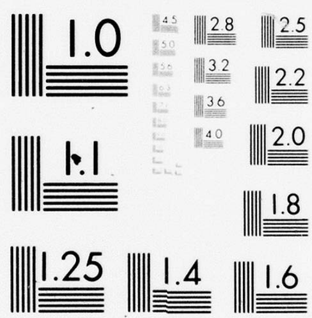

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Until recently the method of integral operators as initiated by S. Bergman and I. N. Vekua has been restricted to the case of elliptic equations and the investigation of steady state phenomena. In these lectures we survey the recent developments on the use of integral operators to investigate equations associated with evolutionary phenomena, in particular parabolic equations, pseudoparabolic equations, and the reduced wave equation in a stratified medium. The topics discussed are transformation operators for partial differential equations, reflection principles and their application, the

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20. Abstract

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Integral Operator Methods in the

Theory of Wave Propagation

and Heat Conduction*

by

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1977

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Abstract

Until recently the method of integral operators as initiated by S. Bergman [3] and I. N. Vekua [75] has been restricted to the case of elliptic equations and the investigation of steady state phenomena. In these lectures we survey the recent developments on the use of integral operators to investigate equations associated with evolutionary phenomena, in particular parabolic equations, pseudoparabolic equations, and the reduced wave equation in a stratified medium. The topics discussed are transformation operators for partial differential equations, reflection principles and their application, the propagation of radio waves around the earth, the propagation of acoustic waves in a spherically stratified medium, low frequency approximations to acoustic scattering problems in a spherically stratified medium, heat conduction in two temperatures, inverse problems in the theory of heat conduction, and Runge's theorem for parabolic equations. Open problems are given at the end of each section.

I. Introduction

Since 1960 a variety of books ([3], [5], [10], [11], [41], [42], [43], [53], [75], [76]), two conference proceedings ([57], [66]), plus chapters in several books on partial differential equations ([4], [6], [34], [40], [47], [78]) have been written on the use of integral operators and/or function theoretic methods in the theory of partial differential equations. Most of this work, until recently, has been concerned with elliptic equations and steady state problems. Recently an analogous approach has been discovered for equations associated with evolutionary phenomena, in particular parabolic equations, pseudoparabolic equations, and the reduced wave equation in a stratified medium. These lectures are devoted to surveying some of these recent developments, and their application. Only a brief outline of proofs will be given, and the reader interested in detailed proofs is referred to the list of references. Although most of the material presented in these lectures has been developed since the appearance of the author's monographs [10] and [11], preliminary versions of some of the results of these lectures can be found in these books. Open problems, of varying difficulty, will be given at the end of each section.

Before proceeding perhaps a few biased words on the role of integral operators in the theory of linear partial differential equations are in order. In view of the large strides that have been made in recent years in developing a general theory of linear partial differential equations, it is sometimes suggested that the theory of boundary value problems for linear partial differential equations is essentially a closed book. That this is far from the case can be attested to by any applied mathematician. More often than not the general theory is not applicable to the particular problem being investigated due to the fact that the operator is not definite, the domain is unbounded, the boundary data is discontinuous, or even that the problem is improperly posed (c.f., Section VIII). In other cases, problems arise due to the need to actually compute a solution rather than "simply" establish its existence. For example, an existence theorem based on solving an integral equation defined over an unbounded three dimensional domain is often of limited use for computational purposes. In order to handle problems such as these, a variety of methods have been developed which are roughly speaking characterized by the fact that, although they are highly effective for the problems they are designed to treat, by their very nature are restricted to rather limited classes of equations. Perhaps the best example of this is the use of generalized double and single layer potentials in the study of partial differential equations with constant coefficients (although this method can be applied to equations with variable coefficients, the practicality of such an approach from the point of view of analytic and numerical approximations is rather limited). From one point of view the use of generalized potential theory can be seen as a branch of the theory of integral operators, i.e., those operators mapping continuous functions onto solutions of linear partial differential equations with constant coefficients. However, in these lectures integral operators will, in general, be viewed in a more restrictive sense, i.e., that in which the integral operator maps solutions of a partial differential equation with constant coefficients onto solutions of a partial differential equation with variable coefficients (or occasionally analytic functions onto solutions of a partial differential equation). Obviously, integral operators in the latter sense and generalized potential theory can often be combined (c.f., Section V), and hence the distinction is not a sharp one. In fact it is perhaps more fruitful to view generalized potential theory and the method of integral operators as complimentary, one dealing with partial differential equations with constant coefficients and the other treating certain classes of partial differential equations with variable coefficients. In view of the above described role of the method of integral operators in the theory of linear partial differential equations, the primary interest is not so much in the integral operator itself, but rather in how it can be used to yield constructive methods for solving boundary value problems arising in various areas of applied mathematics. Of course, as with any fruitful area of mathematics, in the process of achieving the desired aim many other problems of independent interest arise along the way. Hence, although our primary aim in these lectures is to illustrate the use of integral operators in applied mathematics by considering various problems arising in the theory of wave propagation and heat conduction, we shall also find the need to examine such topics as reflection principles and the analytic continuation of solutions to partial differential equations, Runge's theorem for parabolic equations, generalized moment problems, and the completeness of certain systems of entire functions.

II. Transformation Operators for Partial Differential Equations.

In this section we shall construct the integral operators to be used in the next four sections. The term transformation operators is used because of the similarity of our operators to transformation operators for ordinary differential equations (c.f., Appendix 4 of [56]). We first consider the parabolic equation

$$u_{xx} + a(x,t)u_x + b(x,t)u = c(x,t)u_t \quad ; \quad c(x,t) > 0 \tag{2.1}$$

and note that by a change of independent and dependent variables (2.1) can be reduced to an equation of the form

$$u_{xx} + q(x,t)u = u_t \quad . \tag{2.2}$$

Hence we restrict our attention to (2.2) and for the sake of simplicity assume that $q(x,t)$ is an entire function of x and t. We look for solutions of (2.2) defined in a neighborhood of the origin in the form

$$\begin{aligned} u(x,t) &= h(x,t) + \int_{-x}^{x} P(s,x,t)\, h(s,t)ds \\ &= (\underset{\sim}{I} + \underset{\sim}{P})h \end{aligned} \tag{2.3}$$

where $h(x,t)$ is a solution of the heat equation

$$h_{xx} = h_t \quad .$$

Substituting (2.3) into (2.2) and integrating by parts show that (2.3) will be a solution of (2.2) if

$$\begin{aligned} P_{xx} - P_{ss} + q(x,t)P &= P_t \\ P(x,x,t) &= -\frac{1}{2}\int_0^x q(s,t)\,ds \\ P(-x,x,t) &= 0 \quad . \end{aligned} \tag{2.5}$$

A solution of (2.5) can be constructed by iteration (c.f. [11], [12]). Note that since $\underset{\sim}{P}$ is a Volterra operator, $\underset{\sim}{I} + \underset{\sim}{P}$ is invertible. For future applications we note that if $u(0,t) = 0$ then (2.3) can be rewritten as

$$\begin{aligned} u(x,t) &= h(x,t) + \int_0^x [P(s,x,t) - P(-s,x,t)]h(s,t)\,ds \\ &= h(x,t) + \int_0^x \tilde{P}(s,x,t)h(s,t)\,ds. \end{aligned} \tag{2.6}$$

By using the operator $\underset{\sim}{I} + \underset{\sim}{P}$ many properties of the solutions to the heat equation can also be obtained for solutions of (2.2). As an example we have the following theorem:

<u>Theorem 2.1</u>: Let $\{\lambda_n\}$ be a sequence of complex numbers such that

$$\lim_{n\to\infty} \frac{n}{\lambda_n^2} > 0$$

and define

$$h_n^{\pm}(x,t) = e^{\pm i\lambda_n x - \lambda_n^2 t} .$$

Then $\{(I+P)h_n^{\pm}\}$ is complete (with respect to the maximum norm) in any rectangle R for solutions $u(x,t)$ of (2.2).

Proof: Let $u(x,t)$ be a solution of (2.2) in R such that $u(x,t)$ is continuous in $\bar{R}$. We can write $u = (\underset{\sim}{I}+\underset{\sim}{P})h$. Hence it suffices to show that $\{h_n\}$ is a complete family of solutions for $h_{xx} = h_t$ in R. In R, $h(x,t)$ can be approximated by an entire solution $\tilde{h}(x,t)$ of the heat equation ([11], [15]), and $\tilde{h}(x,t)$ depends continuously on its Cauchy data in any thin complex neighborhood of $x = 0$ ([11]). The result now follows from the fact that $\left\{e^{-\lambda_n^2 t}\right\}$ are complete in such a neighborhood if $\lim\limits_{n\to\infty} \frac{n}{\lambda_n^2} > 0$ ([56]).

We now turn our attention to elliptic equations with spherically symmetric coefficients defined in a domain in Euclidean three space $\mathbb{R}^3$. We first consider elliptic equations defined in bounded domains and present a version of R. P. Gilbert's "method of ascent" ([42], [44]). The construction given below is due to D. Colton and R. Kress ([31]). Let D be a bounded starlike domain in $\mathbb{R}^3$ such that $D \subset \{|\underset{\sim}{x}| \leq a\}$ and consider the equation.

$$\Delta_3 u + \lambda^2(1+B(r))u = 0 \qquad (2.7)$$

defined in D where $B(r) \in C^1[0,a]$, $x = |\underset{\sim}{x}|$. We look for a solution of (2.7) in D in the form

$$u(r,\theta,\phi) = h(r,\theta,\phi) + \int_0^r G(r,s;\lambda)h(s,\theta,\phi)ds$$

$$= (\underset{\sim}{I}+\underset{\sim}{G})h \qquad (2.8)$$

where (r,θ,ϕ) denote spherical coordinates and

$$\Delta_3 h = 0 \qquad (2.9)$$

in D. Substituting (2.8) into (2.7) and integrating by parts shows that (2.8) will be a solution of (2.7) provided

$$r^2[G_{rr} + \frac{2}{r} G_r + \lambda^2(1+B(r))G] = s^2[G_{ss} + \frac{2}{s} G_s]$$

$$G(r,r;\lambda) = -\frac{\lambda^2}{2r}\int_0^r s(1+B(s))ds \qquad (2.10)$$

and $G(r,0;\lambda)$ is bounded for $0 \leq r \leq a$. The solution of (2.10) can be found by iteration ([31]). The function $G(r,s;\lambda)$ can in fact be related to the Riemann function for a related hyperbolic equation:

Theorem 2.2: $G(r,s;\lambda) = -(\frac{s}{r})^{1/2}R_3(r,r;s,0)$ where $R(x,y;\xi,\eta)$ is the Riemann function for $R_{xy} + \frac{\lambda^2}{4}(1+B(\sqrt{xy}))R = 0$ where the subscript denotes differentiation with respect to ξ.

Proof: [42], [44].

We now want to present a complimentary operator to $\underset{\sim}{I} + \underset{\sim}{G}$ valid for exterior domains. This operator is due to D. Colton and W. Wendland ([33]) and D. Colton and R. Kress ([30]). Let D be a bounded starlike domain in $\mathbb{R}^3$ and b a real number such that $\{|\underset{\sim}{x}| < b\} \subset D$. Consider (2.7) defined in $\mathbb{R}^3\backslash D$ and assume that $B(r) \in C^1(b,\infty)$, $B(r) = O(e^{-\gamma r^2})$, and $\lambda < 2\gamma b$. We look for a solution of (2.7) in $\mathbb{R}^3\backslash D$ in the form

$$\begin{aligned} u(r,\theta,\phi) &= h(r,\theta,\phi) + \int_r^\infty K(r,s;\lambda)h(s,\theta,\phi)\,ds \\ &= (\underset{\sim}{I}+\underset{\sim}{K})h \end{aligned} \tag{2.11}$$

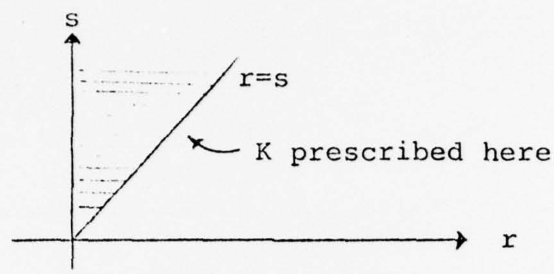
where now $h(r,\theta,\phi)$ is a solution of

$$\Delta_3 h + \lambda^2 h = 0 \tag{2.12}$$

in $\mathbb{R}^3\backslash D$. It is also possible to construct an operator whose domain is solutions of Laplace's equation; however for purposes of application to problems in scattering theory we restrict ourselves here to an operator whose domain is solutions of the Helmholtz equation. Substituting (2.11) into (2.7) and integrating by parts shows that (2.11) will be a solution of (2.7) provided

$$\begin{gathered} r^2[K_{rr} + \frac{2}{r}K_r + \lambda^2(1+B(r))K] = s^2[K_{ss} + \frac{2}{s}K_s + \lambda^2 K] \\ K(r,r;\lambda) = -\frac{\lambda^2}{2r}\int_r^\infty sB(s)\,ds \\ K(r,s;\lambda) = O(e^{-\gamma rs+\lambda^2 s/4\gamma r}) \quad \text{for} \quad b \le r \le s < \infty \ . \end{gathered} \tag{2.13}$$

The solution of (2.13) can be found by iteration ([30]).



Note that if $h(r,\theta,\phi)$ satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r\left(\frac{\partial h}{\partial r} - i\lambda h\right) = 0 \tag{2.14}$$

so does $u(r,\theta,\phi) = (\underset{\sim}{I}+\underset{\sim}{K})h$. We finally observe that since $\underset{\sim}{G}$ and $\underset{\sim}{K}$ are Volterra operators, $\underset{\sim}{I} + \underset{\sim}{G}$ and $\underset{\sim}{I} + \underset{\sim}{K}$ are invertible.

<u>Open Problem</u>: Obtain asymptotic estimates as $\lambda \to +\infty$ for $G(r,s;\lambda)$ and $K(r,s;\lambda)$. Such results would be useful in connection with various problems arising in scattering theory.

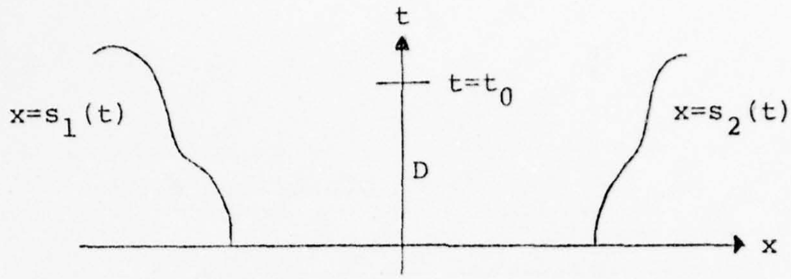
<u>Open Problem</u>: Construct transformation operators for $\Delta_3 u + B(r)u = u_t$ having as domains solutions of the heat equation $\Delta_3 u = u_t$. (For partial progress in this direction see [65]).

III. <u>Reflection Principles and Their Applications</u>.

We first consider the parabolic equation

$$u_{xx} + a(x,t)u_x + b(x,t)u = u_t \tag{3.1}$$

defined in a domain of the form pictured below.



We assume that $u(x,t) \in C^2(D) \cap C^0(\bar{D})$, that $a(x,t)$ and $b(x,t)$ are entire functions of x and t (this assumption can be relaxed), and that $s_1(t)$ and $s_2(t)$ are analytic in a neighborhood of $[0,t_0]$. Define the "reflection" of D across $\sigma : x = s_1(t)$ by

$$D^* = \{(x,t) : 2s_1(t)-s_2(t) < x < s_1(t), \quad 0 < t < t_0\}.$$

Theorem 3.1: Suppose $u(x,t)$ is a solution of (3.1) such that $u(x,t) = f(t)$ on σ where $f(t)$ is analytic in a neighborhood of $[0,t_0]$. Then $u(x,t)$ can be uniquely continued as a solution of (3.1) into $D \cup \sigma \cup D^*$.

Proof ([11], [12], [13]): A change of variables transforms the problem into the case when $a(x,t) = 0$, $s_1(t) = 0$. By constructing a special solution of a non-characteristic Cauchy problem for (3.1) we can further reduce the problem to the case when $f(t) = 0$. From Section II we have

$$u(x,t) = h(x,t) + \int_0^x \tilde{P}(s,x,t)h(s,t)ds$$

where $h(0,t) = 0$. The Theorem now follows from the reflection principle for the heat equation ([80]).

Theorem 3.2: Theorem 3.1 remains valid if the condition $u(x,t) = f(t)$ on σ is replaced by

$$\alpha(t)u(x,t) + \beta(t)u_x(x,t) + \gamma(t)u_t(x,t) = f(t) \quad \text{on} \quad \sigma$$

provided $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $f(t)$ are analytic in a neighborhood of $[0,t_0]$ and $\vec{\mu}(t) = (\beta(t),\gamma(t))$ is never tangent to σ and either never parallel to the x-axis or always parallel to the x-axis.

Proof: [14].

The above reflection principles can be used to extend Theorem 2.1 from the case of solutions of (2.2) defined in a rectangle to the case of solutions of (2.2) defined in a domain with moving boundaries:

Theorem 3.3: The set $\{(\underset{\sim}{I}+\underset{\sim}{P})h_n\}$ is a complete family of solutions to (2.2) in D for

$$1) \quad h_n^{\pm}(x,t) = e^{\pm i\lambda_n x - \lambda_n^2 t}, \quad \lim_{n\to\infty} \frac{n}{\lambda_n^2} > 0,$$

or

$$2) \quad h_n(x,t) = \sum_{k=0}^{[n/2]} \frac{x^{n-2k}t^k}{(n-2k)!k!} .$$

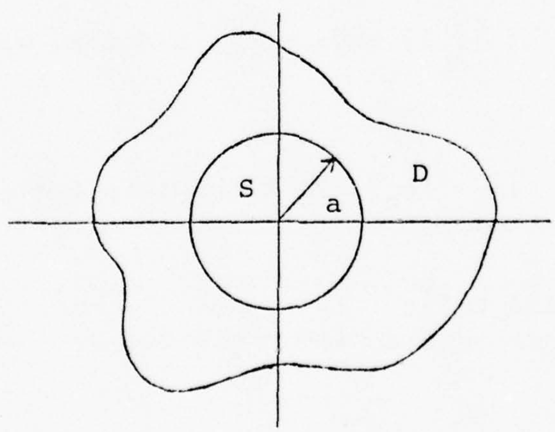
Proof ([11], [16]): Approximate the boundary data by analytic functions, reflect repeatedly across $x = s_1(t)$ and $x = s_2(t)$, and use the fact that the sets 1) and 2) defined above are complete for solutions of the heat equation defined in a rectangle.

For numerical applications of Theorem 3.3 see [8].

We now turn our attention to deriving a reflection principle for solutions to the Helmholtz equation in $\mathbb{R}^n$ which is analogous to the Schwarz reflection principle for harmonic functions vanishing on a portion of a spherical boundary. Our aim is to then use this reflection principle to deduce a continuation theorem connected with the inverse scattering problem for acoustic waves. Let D be a bounded, starlike domain in $\mathbb{R}^n$ containing the ball $S = \{\underset{\sim}{x} : |\underset{\sim}{x}| \leq a\}$ in its interior, and let $u(\underset{\sim}{x}) = u(r,\theta)$, $\theta = (\theta_1, \ldots, \theta_{n-1})$, be a solution of

$$\Delta_n u + \lambda^2 u = 0 \tag{3.2}$$

in $D \setminus S$.



If $u(a,\theta) = 0$, then in $D \setminus S$ we can represent $u(r,\theta)$ in the form ([27])

$$u(r,\theta) = h(r,\theta) + \int_a^r s^{n-3} K(r,s;\lambda) h(s,\theta)\, ds \tag{3.3}$$

where

$$\Delta_n h = 0 \tag{3.4}$$

in $D\setminus S$, $h(a,\theta) = 0$, and $K(r,s;\lambda)$ is a solution of the initial value problem

$$r^2\left[K_{rr} + \frac{(n-1)}{r} K_r + \lambda^2 K\right] = s^2\left[K_{ss} + \frac{(n-1)}{s} K_s\right]$$

$$K(r,r;\lambda) = -\frac{\lambda^2}{4} r^{2-n}(r^2-a^2) \tag{3.5}$$

$$K(r,a;\lambda) = 0 .$$

The solution of (3.5) can be found by iteration ([27]). Hence we have the following theorem:

<u>Theorem 3.4</u>: If $u(r,\theta)$ is a solution of (2.2) in $D\setminus S$ such that $u(a,\theta) = 0$, then $u(r,\theta)$ can be uniquely continued as a solution of (2.2) into $D^* = \left\{(r,\theta) : \left(\frac{a^2}{r},\theta\right) \varepsilon\ D\setminus S\right\}$.

<u>Proof ([27])</u>: This follows from the representation (3.3) and the Schwarz reflection principle for harmonic functions.

Now let $n = 3$ and suppose $u(\underset{\sim}{x}) = u(r,\theta,\phi)$ is a solution of (3.2) in the exterior of a bounded simply connected domain Ω such that $u(r,\theta,\phi)$ satisfies the Sommerfeld radiation condition

$$\lim_{r\to\infty} r(u_r - i\lambda u) = 0 . \tag{3.6}$$

Then at infinity

$$u(r,\theta,\phi) = \frac{e^{i\lambda r}}{r} f(\theta,\phi;\lambda) + 0\left(\frac{1}{r}\right) . \tag{3.7}$$

Suppose that on $\partial\Omega$ we have $u(r,\theta,\phi) = e^{i\lambda r\cos\theta}$. The inverse scattering problem is to determine Ω from a knowledge of $f(\theta,\phi;\lambda)$. It is known that $f(\theta,\phi;\lambda)$ uniquely determines Ω ([55]). From $f(\theta,\phi;\lambda)$ we can determine $u(r,\theta,\phi)$ outside the smallest ball S containing Ω in its interior (c.f. [60]). Hence the problem is to continue $u(r,\theta,\phi)$ across ∂S and determine the locus of points such that $u(r,\theta,\phi) - e^{i\lambda r\cos\theta} = 0$. This can be done (in theory!) by using Theorem 3.4 (after a reduction to the case when $u(r,\theta,\phi) = 0$ on ∂S - c.f. [27]) in conjunction with R. P. Gilbert's "envelope method" (c.f. [10], [41], [42]) as applied to harmonic functions.

Definition 3.1: Let $f(z)$ be an entire function of exponential type. Then the indicator diagram of $f(z)$ is the interior of the convex hull of the singularities of the Laplace transform of $f(z)$ (The indicator diagram can also be defined in terms of the growth of $f(z)$ along rays passing through the origin - c.f. [56]).

Theorem 3.5: Let $u(r,\theta,\phi) = u(r,\theta)$ be axially symmetric (which implies $f(\theta,\phi) = h(\cos\theta)$). Define

$$F(z) = \int_{-1}^{1} h(\xi) \frac{(1+4z^2)d\xi}{(1-4iz\xi-4z^2)^{3/2}} \; ; \; |z| < \frac{1}{2} .$$

(Note that $h(\xi) = \sum_{n=0}^{\infty} a_n P_n(\xi)$ $F(z) = 2 \sum_{n=0}^{\infty} a_n (2iz)^n$). Then $F(z)$ can be continued to an entire function of exponential type. If I is the indicator diagram of $F(z)$, then $u(r,\theta)$ is regular in the exterior of $I \cup \bar{I}$, where the bar denotes complex conjugation.

Proof: [10], [24].

Remark: In [10] and [24] it was not possible to exclude the possibility that $u(r,\theta)$ was singular along the axis $\theta = 0$ or $\theta = \pi$ since at that time Theorem 3.4 was unavailable. However, if in the proofs of [10] and [24] one uses Theorem 3.4 instead of Lewy's reflection principle as applied to the axially symmetric Helmholtz equation, one arrives at Theorem 3.5.

For another approach to the inverse scattering problem see [79].

Open Problem: Derive an analogue of Theorem 3.5 when Ω is no longer a bounded domain.

Open Problem: In Theorem 3.2 remove the restriction that $\vec{\mu}(t)$ is either never parallel to the x-axis or always parallel to the x-axis.

IV. The Propagation of Radio Waves Around the Earth.

In this section we shall show how the transformation operators obtained in Section II can be used to construct approximate solutions to the Fock-Leontovich equations describing radiowave propagation around the earth under the assumption of a spherically stratified atmosphere but ignoring the effect of the ionosphere ([38]). We assume that a vertical electric dipole is situated on the surface of the earth, and without loss of generality we assume that this point is the north pole. Let (x,y) denote the point of observation of the electromagnetic field, where x is the (normalized) distance along the surface of the earth from the north pole to a point directly below the observation point, and y is the (normalized) distance of the observation point to the surface of the earth. Let a denote the radius of the earth, k the wave number, and $w(x,y)$ the (normalized) Hertz potential (*For precise definitions* the reader is referred to [38]). Then under the assumption of a perfectly conductive earth and using the fact that ka is very large, we are led to the following initial-boundary value problem for $w(x,y)$:

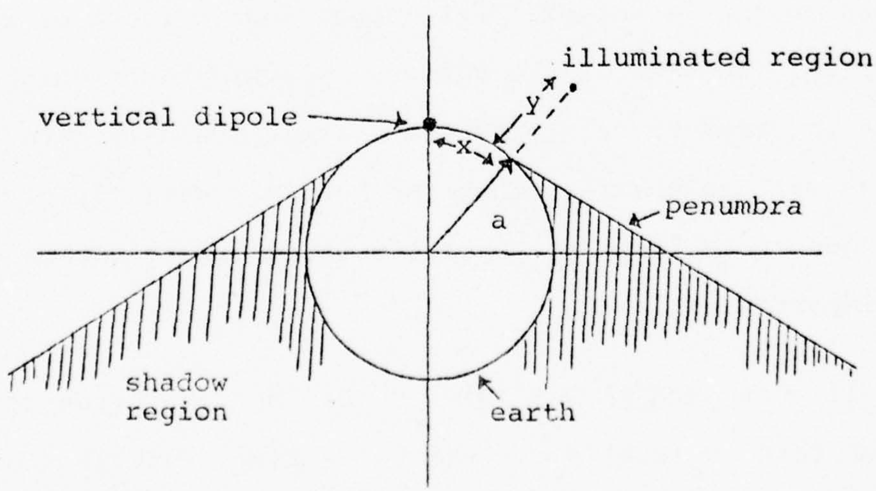
$$w_{yy} + iw_x + y(1+g(y))w = 0 \tag{4.1a}$$

$$w_y(x,0) = 0; \quad 0 < x < x_0 \tag{4.1b}$$

$$w(x,y_0) = \frac{2}{\sqrt{x}} \exp(iy_0{}^2/4x); \quad 0 < x < x_0 \tag{4.1c}$$

$$\lim_{x\to 0}\left[w(x,y) - \frac{2}{\sqrt{x}} \exp(iy^2/4x)\right] = 0; \quad 0 < y < y_0 \tag{4.1d}$$

where $g(y)$ is a real-valued slowly varying function (in our case assumed to be a polynomial) related to the refractive index of the atmosphere and x_0, y_0 are positive constants such that $x < x_0$, $y < y_0$, $y_0 >> x_0^2$. (4.1a) is the Fock-Leontovich equation, (4.1b) reflects the fact that the earth is perfectly conductive, (4.1c) is a consequence of the Fresnel reflection law and geometrical optics, and (4.1d) is due to the presence of a vertical electric dipole at the north pole. For a full discussion of the derivation of (4.1a)-(4.1d) see [38] and [17]. The advantage of studying the equations (4.1a)-(4.1d) instead of the usual boundary value problem associated with the Helmholtz equation is that it permits the investigation of not only the illuminated region $(y >> x^2)$ and the shadow region $(y << x^2)$ but also intermediate cases, namely the region of the penumbra.



We look for a solution of (4.1a)-(4.1d) in the region $D = \{(x,y) : 0 < x < x_0,\ 0 < y < y_0\}$ in the form

$$w(x,y) = \frac{2}{\sqrt{x}} \exp(iy^2/4x)$$

$$+ \frac{2}{\sqrt{x}} \int_0^y [P(s,y)+P(-s,y)]\exp(is^2/4x)ds + u(x,y) \tag{4.2}$$

where $P(s,y)$ is the kernel $P(s,y,t)$ defined in Section II (which in our case is independent of t) and $u(x,y) \in C^2(D) \cap C^0(\bar{D})$ is a solution of (4.1a) such that $u_y(x,0) = 0$. We now want to determine the remaining initial-boundary data for $u(x,y)$ and derive a method for approximating $u(x,y)$ as well as the term

$$\frac{2}{\sqrt{x}} \int_0^y [P(s,y)+P(-s,y)]\exp(is^2/4x)ds \tag{4.3}$$

appearing in (4.2). Note that this last approximation is non-trivial due to the factor of $2/\sqrt{x}$ appearing in front of the integral sign. In view of the various approximations which need to be made in order to determine the initial-boundary data for $u(x,y)$, as well as the fact that the boundary data (4.1c) is an approximation to begin with, the following a priori estimate is of basic importance:

Theorem 4.1: Let $u(x,y) \in C^2(D) \cap C^0(\bar{D})$ be a solution of (4.1a) in D such that $u_y(x,0) = 0$, $u(x,y_0) = f(x)$, $u(0,y) = h(y)$. Then there exists a positive constant M such that

$$\int_0^{y_0}\int_0^{x_0}\left|\int_0^s u(\tau,\eta)d\tau\right|^2 ds d\eta \le M\left[\int_0^{x_0}|f(s)|^2 ds + \int_0^{y_0}|h(\eta)|^2 d\eta\right] .$$

Proof ([17]): Define

$$u_1(x,y) = \int_0^x u(\tau,y)d\tau - \int_0^x f(\tau)d\tau.$$

Then $u_1(x,y)$ satisfies a non-homogeneous version of (4.1a) with homogeneous initial-boundary data. The result now follows by applying standard eigenfunction expansion methods. Note that due to the fact that in general $f'(x) \not\in L^2(0,x_0)$, it is not possible (by these methods) to replace the weighted L^2-norm by a L^2 norm on the left hand side of the inequality in Theorem 4.1.

We now proceed to approximate (4.3) and the initial-boundary data $f(x)$ and $h(y)$ as defined in Theorem 4.1. From a result on asymptotic expansions due to Erdélyi ([36]) it is possible to show that

$$\frac{2}{\sqrt{x}} \int_0^y [P(s,y)+P(-s,y)]\exp(is^2/4x)ds = 2\sqrt{2\pi}(1+i)P(0,y)+R(x,y) \tag{4.4}$$

where $|R(x,y)| \leq \text{constant}\cdot\max\limits_D |P_s(s,y)-P_s(-s,y)|$. By using this result and the asymptotic properties of Fresnel integrals (after approximating $P(s,y)+P(-s,y)$ by a polynomial) one can show that to within $0(4x_0/y_0{}^2)^{1/2}$ we have

$$\begin{aligned} f(x) &= P_1(x) + \sqrt{x}\exp(iy^2/4x)P_2(x) \\ h(y) &= P_3(y) \end{aligned} \tag{4.5}$$

where $P_1(x)$, $P_2(x)$ and $P_3(y)$ are polynomials. A similar analysis allows us to approximate (4.3) for $0 \le x \le x_0$, $0 \le y \le y_0$. For details the reader is referred to [17].

Our task now is to approximate $u(x,y)$ satisfying (4.1a) along with the initial-boundary data $u_y(x,0) = 0$, $u(x,y_0) = f(x)$, $u(0,y) = h(y)$ where $f(x)$ and $h(y)$ are given by (4.5). To this end we shall again make use of the transformation operators constructed in Section II.

Theorem 4.2: Assume $x_0 \le y_0^2$ and let $\lambda_n = \dfrac{2n\pi}{y_0^2}$, $\mu_n = (n-1/2)\dfrac{\pi}{y_0}$. Let

$$h_{2n}(x,y) = \cos\sqrt{\lambda_n}\, y\, e^{-i\lambda_n x}, \quad n = \pm 1, \pm 2, \ldots$$
$$h_{2n+1}(x,y) = \cos \mu_n y\, e^{-i\mu_n^2 x}, \quad n = 1,2,\ldots \quad .$$

Then with respect to the norm defined in Theorem 4.1, the set $\{u_n\}$ where

$$u_n(x,y) = h_n(x,y) + \int_0^y [P(s,y)+P(-s,y)]h_n(s,y)\,ds$$

is a complete set of solutions to (4.1a) satisfying (4.1b).

Proof ([17]): As in Theorem 2.1, it sufficies to show that the set $\{h_n\}$ is complete for solutions of $h_{yy}+ih_x = 0$ satisfying $h_y(x,0) = 0$. But this follows from Theorem 4.1 and Levinson's result that $\left\{e^{-i\lambda_n x}\right\}$ is complete in $L^2(0,x_0)$ if $|\lambda_n| \leq \frac{2\pi}{x_0}(|n| + \frac{1}{4})$ (c.f. [56]).

To approximate $u(x,y)$ we now set

$$u^N(x,y) = \sum_{k=0}^{N} a_k u_k(x,y) \tag{4.6}$$

for a given integer N and minimize (in $\mathbb{R}^{2N}$) the quadratic functional

$$\begin{aligned} Q(a_1,\ldots,a_N) &= \left|\left|u^N(x,y_0)-f(x)\right|\right|^2_{L^2(0,x_0)} \\ &+ \left|\left|u^N(0,y)-h(y)\right|\right|^2_{L^2(0,y_0)} . \end{aligned} \tag{4.7}$$

<u>Open Problem</u>: Treat the case when the earth is no longer perfectly conductive. (The generalization is non-trivial!)

<u>Open Problem</u>: Treat the case when $y_0 = \infty$, i.e. avoid the use of the geometric optics approximation (4.1c).

V. <u>The Propagation of Acoustic Waves in a Spherically Stratified Medium</u>.

In this section we shall show how the transformation operators constructed in Section II can be used in the investigation of acoustic scattering problems in a spherically stratified medium. We first consider the case when the incoming wave is scattered by the presence of a spherically stratified quasi-homogeneous medium, but in the absence of any obstacle. If we denote the velocity potential by $u(\underset{\sim}{x})$ and the velocity potential of the scattered wave by $u_s(\underset{\sim}{x})$ (factoring out a term of the form $e^{-i\omega t}$) we arrive at the following set of equations for the determination of $u_s(\underset{\sim}{x})$:

$$\begin{aligned} u(\underset{\sim}{x}) &= e^{i\lambda z} + u_s(\underset{\sim}{x}) \\ \Delta_3 u + \lambda^2(1+B(r))u &= 0 \quad \text{in} \quad \mathbb{R}^3 \\ \lim_{r\to\infty} r\left(\frac{\partial u_s}{\partial r} - i\lambda u_s\right) &= 0 \end{aligned} \tag{5.1}$$

where λ is the wave number, $x = (x,y,z)$, $r = |\underset{\sim}{x}|$, and $B(r) = \left(\frac{c_0}{c(r)}\right)^2 - 1$ where $c(r)$ is the speed of sound and $c_0 = \lim_{r\to\infty} c(r)$. We make the assumption that $B(r) = O(e^{-\gamma r^2})$ and $0 < \lambda < 2\gamma(\sqrt{2}-1)$. We look for a solution of (5.1) in the following form: For $r \leq 1$ we represent $u(\underset{\sim}{x})$ as

$$\begin{aligned} u(\underset{\sim}{x}) &= \sum_{n=0}^{\infty} a_n u_n(r) P_n(\cos\theta) \\ u_n(r) &= (\underset{\sim}{I}+\underset{\sim}{G}) r^n \end{aligned} \tag{5.2}$$

where $\underset{\sim}{I} + \underset{\sim}{G}$ is the transformation operator defined in Section II, $P_n(\cos\theta)$ is Legendre's polynomial, and the constants a_n are to be determined. For $r \geq 1$ we represent $u(\underset{\sim}{x})$ as

$$u(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{K})\left[e^{i\lambda z} + \sum_{n=0}^{\infty} b_n h_n^{(1)}(\lambda r)P_n(\cos\theta)\right] \tag{5.3}$$

where $(\underset{\sim}{I}+\underset{\sim}{K})$ is the transformation operator defined in Section II, $h_n^{(1)}(\lambda r)$ is a spherical Hankel function, and the constants b_n are to be determined. If we now use Sonine's formula to expand $e^{i\lambda z}$ and require $u(\underset{\sim}{x})$ and its first derivatives to agree at $r = 1$ we are led to the following algebraic system for the determination of the constants a_n and b_n:

$$\begin{aligned} a_n u_n(1) - b_n\left[(\underset{\sim}{I}+\underset{\sim}{K})h_n^{(1)}(\lambda r)\right]_{r=1} &= (2n+1)i^n\left[(\underset{\sim}{I}+\underset{\sim}{K})j_n(\lambda r)\right]_{r=1} \\ a_n u_n'(1) - b_n\left[\frac{d}{dr}(\underset{\sim}{I}+\underset{\sim}{K})h_n^{(1)}(\lambda r)\right]_{r=1} &= (2n+1)i^n\left[\frac{d}{dr}(\underset{\sim}{I}+\underset{\sim}{K})j_n(\lambda_r)\right]_{r=1} \end{aligned} \tag{5.4}$$

where $j_n(\lambda r)$ denotes a spherical Bessel function. From (5.4) it is now possible to deduce by using Crammer's rule and uniform estimates for Bessel and Hankel functions that

$$\begin{aligned} \left|\int_1^{\infty} K(1,s;\lambda)j_n(\lambda s)ds\right| &= O\left[\left(\frac{\lambda}{2\beta}\right)^n\right] \\ \left|\int_1^{\infty} K(1,s;\lambda)h_n^{(1)}(\lambda s)ds\right| &= O\left[\frac{\Gamma(n+1/2)2^n}{n\lambda^n}\right] \end{aligned} \tag{5.5}$$

where $\beta = \gamma - \lambda^2/4\gamma$ and $K(r,s;\lambda)$ is the kernel of the operator $\underset{\sim}{I} + \underset{\sim}{K}$. These estimates imply that

$$a_n = O\left(n\left(\frac{\lambda}{2\beta}\right)^n\right)$$
$$b_n = O\left[\frac{n^2}{n!}\left(\frac{\lambda^2}{4\beta}\right)^n\right] \tag{5.6}$$

which imply that the series representations for $u(\underset{\sim}{x})$ converge and satisfy (5.1). The uniqueness of the solution follows from the fact that for λ real, $u_n(r)$ and $(\underset{\sim}{I}+\underset{\sim}{K})h_n^{(1)}(\lambda r)$ are linearly independent solutions of

$$y'' + \frac{2}{r}y' + \left[\lambda^2(1+B(r)) - \frac{n(n+1)}{r^2}\right]y = 0. \tag{5.7}$$

For details of the above calculations the reader is referred to [26] and [30].

We now turn our attention to the same problem considered above, except that in addition to the spherically stratified medium there is a "hard" obstacle D present. We assume that D is bounded, starlike, and has smooth boundary ∂D with outward pointing unit normal ν. In this case the equations for determining the velocity potential $u(\underset{\sim}{x})$ become

$$u(\underset{\sim}{x}) = e^{i\lambda z} + u_s(\underset{\sim}{x})$$
$$\Delta_3 u + \lambda^2(1+B(r))u = 0 \quad \text{in} \quad \mathbb{R}^3\backslash D$$
$$\frac{\partial u}{\partial \nu} = 0 \quad on \quad \partial D \tag{5.8}$$
$$\lim_{r\to\infty} r\left(\frac{\partial u_s}{\partial r} - i\lambda u_s\right) = 0 \quad .$$

We again assume that $B(r) = O(e^{-\gamma r^2})$ and restrict ourselves to values of λ such that $\lambda < 2\gamma b$ where $\{|\underset{\sim}{x}| < b\} \subset D$. We look for a solution in the form

$$u(r,\theta,\phi) = h(r,\theta,\phi) + \int_r^\infty K(r,s;\lambda)h(s,\theta,\phi)ds \tag{5.9}$$

where $K(r,s;\lambda)$ is the kernel of the transformation operator $\underset{\sim}{I} + \underset{\sim}{K}$ and $h(\underset{\sim}{x})$ is a solution of

$$\Delta_3 h + \lambda^2 h = 0 \tag{5.10}$$

in $\mathbb{R}^3 \setminus D$ of the form

$$\begin{aligned} h(\underset{\sim}{x}) &= e^{i\lambda z} + h_s(\underset{\sim}{x}) \\ \lim_{r\to\infty} r\left(\frac{\partial h_s}{\partial r} - i\lambda h_s\right) &= 0 \quad . \end{aligned} \tag{5.11}$$

Following Ursell ([74]; see also [50]) we represent $h_s(\underset{\sim}{x})$ in the form

$$h_s(x) = \int_{\partial D} \mu(\underset{\sim}{\xi}) G(\xi;\underset{\sim}{x};\lambda) d\omega_\xi \tag{5.12}$$

where $\mu(\underset{\sim}{\xi})$ is a potential to be determined and $G(\xi;x;\lambda)$ is a fundamental solution of the Helmholtz equation in the exterior of $\{|\underset{\sim}{x}| \leq b\}$ satisfying the Sommerfeld radiation condition, and on $|\underset{\sim}{x}| = b$

$$\left(\frac{\partial}{\partial r} + \alpha\right) G = 0 \tag{5.13}$$

where $\alpha = e^{i\delta}$, $0 < \delta < \pi$. The function $G(\xi;x;\lambda)$ can be constructed by separation of variables and is introduced to avoid the problem of the non-invertibility of the integral equation associated with scattering problems for λ an eigenvalue of the interior Dirichlet problem of the Helmholtz equation ([74]). If we now substitute $e^{i\lambda z} + h_s(\underset{\sim}{x})$ into (5.9) with $h_s(\underset{\sim}{x})$ defined by (5.12), interchange orders of integration and let $\underset{\sim}{x}$ tend to ∂D, we arrive at a Fredholm integral equation for the determination of $\mu(\underset{\sim}{\xi})$ of the form

$$\frac{1}{2\pi} f(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{T}(\lambda))\mu \quad ; \quad \underset{\sim}{x} \in \partial D \tag{5.14}$$

where $f(\underset{\sim}{x}) = \frac{\partial}{\partial \nu} (\underset{\sim}{I}+\underset{\sim}{K})e^{i\lambda z}$ and $\underset{\sim}{T}(\lambda)$ is a compact integral operator on the Banach space $C^0(\partial D)$ ([11], [33]).

<u>Theorem 5.1</u>: Let $0 < \lambda < 2\gamma b$. Then in $C^0(\partial D)$, $(\underset{\sim}{I}+\underset{\sim}{T}(\lambda))^{-1}$ exists.

<u>Proof ([33])</u>: Since $\underset{\sim}{K}$ is a Volterra operator, $(\underset{\sim}{I}+\underset{\sim}{K})^{-1}$ exists, and hence if $(\underset{\sim}{I}+\underset{\sim}{T}(\lambda))\mu = 0$ we can conclude that $(\underset{\sim}{I}+\underset{\sim}{T}_0(\lambda))\mu = 0$ where $\underset{\sim}{T}_0 = \underset{\sim}{T}\Big|_{B(r)=0}$. From the choice of $G(\underset{\sim}{\xi};\underset{\sim}{x};\lambda)$ we can now conclude that $\mu(\xi) = 0$, and hence by the Fredholm Alternative $(\underset{\sim}{I}+\underset{\sim}{T}(\lambda))^{-1}$ exists.

Note that one advantage of the integral operator method outlined above over the straight forward use of integral equations is that the resulting integral equation is defined over a bounded two dimensional region instead of an unbounded three dimensional region.

Open Problem: Investigate problem (5.8) in the case when D is no longer starlike with respect to the origin. For a preliminary investigation of this problem see [29].

Open Problem: Viewing $\tilde{I}+\tilde{T}(\lambda)$ as an operator valued analytic function of λ, determine the nearest singularities to the origin of $(\tilde{I}+\tilde{T}(\lambda))^{-1}$. Such a result would lead to new constructive methods for solving (5.8) obtained by expanding $(\tilde{I}+\tilde{T}(\lambda))^{-1}$ in powers of λ and determining the coefficients recursively.

For a survey of the results of this section in the case when $B(r)$ has compact support see [28].

VI. Low Frequency Approximations to Acoustic Scattering Problems in a Spherically Stratified Medium.

In this section we shall consider the problem of approximating the solutions of the problems discussed in the previous section in the case when the wave number λ is small. In the case of a homogeneous medium, the problem of low frequency approximations to scattering problems has been extensively investigated by Ralph Kleinman and his co-workers (c.f. [51], [52]). We first consider the problem of scattering by a spherically stratified medium in the absence of an obstacle, i.e. to determine the velocity potential $u(\tilde{x})$ from the equation

$$\begin{aligned} u(\tilde{x}) &= e^{i\lambda z} + u_s(\tilde{x}) \\ \Delta_3 u + \lambda^2(1+B(r))u &= 0 \quad \text{in } \mathbb{R}^3 \qquad (6.1) \\ \lim_{r\to\infty} r\left(\frac{\partial u_s}{\partial r} - i\lambda u_s\right) &= 0 \end{aligned}$$

where we now assume that λ, in some sense, is small. In this case a classical method for solving (6.1) is to reformulate the problem as the integral equation

$$u(\underset{\sim}{x}) = e^{i\lambda z} + \lambda^2 \iint_{\mathbb{R}^3} \left[\frac{1}{4\pi} \frac{e^{i\lambda|\underset{\sim}{x}-\underset{\sim}{\xi}|}}{|\underset{\sim}{x}-\underset{\sim}{\xi}|}\right] B(|\underset{\sim}{\xi}|)\, u(\underset{\sim}{\xi})\, d\underset{\sim}{\xi} . \tag{6.2}$$

It can be shown (c.f. [35]) that (6.2) can be solved by iteration, yielding the first and higher order Born approximations, provided

$$\lambda^2 \int_0^\infty r|B(r)|dr < 1. \tag{6.3}$$

Recent investigations in this direction have been made by Rorres ([63]) and Ahner ([1]). We propose to study (6.1) by the method of integral operators, and shall show that for λ satisfying (6.3) the Born approximations are recovered, whereas our method also yields approximations in the case when (6.3) is no longer valid. As in the previous section we assume that $B(r) = O(e^{-\gamma r^2})$ and

$$0 < \lambda < 2\gamma(\sqrt{2}-1) \quad . \tag{6.4}$$

We restrict ourselves to $r \geq 1$ and recall from Section V that in this case (c.f. (5.3))

$$u(\underset{\sim}{x}) = (\underset{\sim}{I}+\underset{\sim}{K})\left[e^{i\lambda z} + \sum_{n=0}^{\infty} b_n h_n^{(1)}(\lambda r) P_n(\cos\theta)\right]; \; r \geq 1 \tag{6.5}$$

where $b_n = b_n(\lambda)$ is given by (c.f. (5.4))

$$(\underset{\sim}{I}+\underset{\sim}{K})[(2n+1)i^n j_n(\lambda r)+b_n(\lambda)h_n^{(1)}(\lambda r)]_{r=1}u_n'(1) \tag{6.6}$$

$$= u_n(1)\frac{d}{dr}(\underset{\sim}{I}+\underset{\sim}{K})[(2n+1)i^n j_n(\lambda r)+b_n(\lambda)h_n^{(1)}(\lambda r)]_{r=1} .$$

The quantities $(\underset{\sim}{I}+\underset{\sim}{K})e^{i\lambda z}$ and $(\underset{\sim}{I}+\underset{\sim}{K})h_n^{(1)}(\lambda r)$ can be readily approximated, for small and moderately large value of λ, by truncating the iterative process used to construct the kernel $K(r,s;\lambda)$ of the operator $\underset{\sim}{K}$ (c.f. [30]). Hence the problem is to use (6.6) to approximate $b_n(\lambda)$. From (6.6) we have that $b_n(\lambda)$ is an analytic function of λ in a neighborhood of the origin and has a Taylor expansion of the form

$$b_n(\lambda) = \lambda^{2n+3}\sum_{k=0}^{\infty} b_{nk}\lambda^k \quad . \tag{6.7}$$

Hence a low frequency approximation to $b_n(\lambda)$ is given by $b_n(\lambda) \approx b_{n0}\lambda^{2n+3}$. For larger values of λ such that (6.4) is still valid $b_n(\lambda)$ can be approximated by using (6.6) in conjunction with the asymptotic expansions of $j_n(\lambda r)$ and $h_n^{(1)}(\lambda r)$. Returning now to the low frequency approximation $b_n(\lambda) \approx b_{n_0}\lambda^{2n+3}$ we note that (c.f. Section II)

$$G(1,s;\lambda) = -\frac{\lambda^2}{2}\int_0^1 s(1+B(s))ds + O(\lambda^4)$$

$$K(1,s;\lambda) = -\frac{\lambda^2}{2}\int_1^{\infty} sB(s)ds + O(\lambda^4) \tag{6.8}$$

and hence from (6.6) we can deduce that

$$b_{n0} = \frac{i^{n+1}}{2n+1} \left(\frac{2^n n!}{(2n)!}\right)^2 \int_0^\infty s^{2n+2} B(s)\,ds \quad . \tag{6.9}$$

This result agrees with the first Born approximation (c.f. [59]) in the case when λ satisfies (6.3).

We now briefly consider the problem of low frequency approximations to acoustic scattering problems in a spherically stratified medium when an obstacle is present. For the sake of simplicity we restrict ourselves to the case when the obstacle is a "soft" sphere of radius one centered at the origin and $B(r) = 0$ for $r \geq a \geq 1$. More general obstacles as well as the case when $B(r)$ no longer has compact support can also be treated with similar (although more complicated) results (c.f. [28]). In our simple case the relevant equations are

$$\begin{gathered} u(\underset{\sim}{x}) = e^{i\lambda z} + u_s(\underset{\sim}{x}) \\ \Delta_3 u + \lambda^2(1+B(r))u = 0 \quad \text{in} \quad \mathbb{R}^3 \backslash \Omega \\ \lim_{r\to\infty} r\left(\frac{\partial u_s}{\partial r} - i\lambda u_s\right) = 0 \end{gathered} \tag{6.10}$$

where $\Omega = \{\underset{\sim}{x} : |\underset{\sim}{x}| \leq 1\}$. The velocity potential $u_s(\underset{\sim}{x})$ of the scattered wave can be found in the form

$$u_s(\underset{\sim}{x}) = \sum_{n=0}^{\infty} b_n(\lambda)(\underset{\sim}{I}+\underset{\sim}{K})h_n^{(1)}(\lambda r)P_n(\cos\theta) \tag{6.11}$$

where $b_n(\lambda)$ is determined by the method of separation of variables ([11], [25]). It turns out that $b_n(\lambda)$ is an analytic function of λ in a neighborhood of the origin and has a Taylor expansion of the form

$$b_n(\lambda) = \lambda^{2n+1} \sum_{k=0}^{\infty} b_{nk}\lambda^k \ . \tag{6.12}$$

Note that in the present case $b_n(\lambda)$ has a zero of order $2n+1$ at the origin, whereas in the case of scattering without the presence of an obstacle $b_n(\lambda)$ had a zero of order $2n+3$ (c.f. (6.7)). In (6.12) it can be seen after a short calculation ([11], [25]) that b_{n0} is independent of $B(r)$ and

$$\gamma_n^{\,0} + b_{n1}\gamma_n^{\,1} = \int_1^a B(s)[s^{2n+2} + s^{-2n} - 2s]ds \tag{6.13}$$

where $\gamma_n^{\,0}$ and $\gamma_n^{\,1}$ are known constants (independent of $B(r)$). Hence for low frequencies the scattering due to the obstacle dominates the scattering due to the inhomogeneous medium and the first low frequency approximation that takes the inhomogeneous medium into account is $b_n(\lambda) \approx \lambda^{2n+1}(b_{n0}+b_{n1}\lambda)$.

The analysis in this section can also be used to investigate the inverse scattering problem of determining $B(r)$ from a knowledge of the far field pattern defined by

$$f(\theta;\lambda) = \lim_{r\to\infty} re^{-i\lambda r} u_s(\underset{\sim}{x})$$
$$= \frac{1}{\lambda} \sum_{n=0}^{\infty} (-i)^{n+1} b_n(\lambda) P_n(\cos\theta). \tag{6.14}$$

We restrict ourselves to the case of problem (6.10).

Theorem 6.1: The set $\left\{r^{2n+2}+r^{-2n}-2r\right\}_{n=0}^{\infty}$ is complete in $L^2[1,a]$.

Proof: [11], [25].

If we recall now the fact that $B(r) = \left(\frac{c_0}{c(r)}\right)^2 - 1$ where $c(r)$ is the speed of sound and $c_0 = \lim_{r\to\infty} c(r)$ is assumed known, we have from (6.13) and Theorem 6.1 the following result:

Corollary: *For problem (6.10) the far field pattern* $f(\theta;\lambda)$, $0 \le \theta \le \pi$, $0 < \lambda < \lambda_0$ *(where* λ_0 *is an arbitrary positive number) uniquely determines the speed of sound in the inhomogeneous medium.*

By orthonormalizing the set $\left\{r^{2n+2}+r^{-2n}-2r\right\}_{n=0}^{\infty}$ in $L^2[1,a]$ and using (6.13) to compute the Fourier coefficients we can also obtain approximations to $B(r)$. However, since $\gamma_n^{\ 1}$ tends to infinity as n tends to infinity this approximation procedure is unstable.

Open Problem: Give a stable method for constructing $B(r)$ from $f(\theta;\lambda)$.

Open Problem: Consider the problems treated in the last two sections for the case when the source is located a finite distance from the scattering obstacle instead of being a plane wave coming in from infinity.

VII. Heat Conduction in Two Temperatures.

In a paper on continuum thermodynamics by Gurtin and Williams ([46]) it was shown that under certain physically reasonable hypotheses one could consider heat conduction being governed by one temperature whereas heat supply by another. It was then shown that for an extremely general class of simple materials these two temperatures turn out to be equal. However, in the case of a non-simple material, in particular one in which the thermodynamic quantities depend on the conductive temperature and its first two spatial derivatives, Chen and Gurtin showed that this was no longer the case ([19]). In particular for an isotropic material the linearized version of the energy equation takes the form

$$c \frac{\partial \phi}{\partial t} = k\Delta\phi + ca \frac{\partial}{\partial t} \Delta\phi + q(\underset{\sim}{x},\underset{\sim}{t}) \qquad (7.1)$$

where $\phi(\underset{\sim}{x},\underset{\sim}{t})$ is the conductive temperature, c the specific heat, k the conductivity, and a is the temperature descrepency relating the conductive temperature to the thermodynamic temperature by the relation

$$\theta = \phi - a\Delta\phi \ . \tag{7.2}$$

From the second law of thermodynamics we have the fact that $a \geq 0$. (7.1) is a particular example of an equation of pseudo-parabolic or Sobolev type. Such equations have been the focus of a considerable amount of interest in recent years, and the reader is referred to [67] and [68] and the references contained in thses papers for further information. In this section we shall restrict ourselves to (7.1) and for the sake of simplicity set all the constants appearing in (7.1) equal to unity. Our primary concern is the solution of initial-boundary value problems for (7.1) and we first consider the case of one space variable. We begin by introducing the idea of a Riemann function. Let L and M denote the operators defined by

$$\begin{aligned} L[u] &\equiv u_{xxt}-u_t+u_{xx} \\ M[v] &= v_{xxt}-v_t-v_{xx} \end{aligned} \tag{7.3}$$

respectively. Then the Riemann function $v(x,t;\xi,\tau)$ for $L[u] = 0$ is defined by

$$\begin{aligned} M[v] &= 0 \\ v_x(\xi,t;\xi,\tau) &= 1-e^{(t-\tau)} \\ v(\xi,t;\xi,\tau) &= 0 \\ v(x,\tau;\xi,\tau) &= 0 \end{aligned} \tag{7.4}$$

and can be constructed by iteration ([10], [21]). If $u(x,t)$ is a solution of

$$\begin{aligned} L[u] &= q(x,t) \\ u(0,t) &= 0 \\ u_x(0,t) &= g(t) \\ u(x,0) &= 0 \end{aligned} \tag{7.5}$$

where $q(x,t)$ is continuous and $g(t)$ continuously differentiable then

$$\begin{aligned} u(x,t) = &\int_0^t [g'(\tau)v_t(0,\tau;x,t)+g(\tau)v_t(0,\tau;x,t)]d\tau \\ &+ \int_0^t \int_0^x q(\xi,\tau)v_t(\xi,\tau;x,t)d\xi d\tau \ . \end{aligned} \tag{7.6}$$

Now suppose we want to construct a solution to the initial-boundary value problem

$$\begin{aligned} L[u] &= q(x,t) \\ u(0,t) &= 0 \\ u(x,0) &= 0 \\ u(a,t) &= 0 \quad . \end{aligned} \tag{7.7}$$

(We note that the case of non-homogeneous initial-boundary data can easily be reduced to this case). From (7.6) we have the following integral equation for $g(t) = u_x(0,t)$:

$$\gamma(t) = g(t)v_t(0,t;a,t) + \int_0^t [v_t(0,\tau;a,t)-v_{tt}(0,\tau;a,t)]g(\tau)d\tau \tag{7.8}$$

where $\gamma(t)$ is a known function.

<u>Lemma 7.1</u>: $v_t(0,t;a,t) \neq 0$.

<u>Proof([10],[21])</u>: From (7.4) we have that $\mu(x) = v_t(x,t;a,t)$ satisfies $\mu_{xx}-\mu = 0$. But $\mu(a) = 0$ and hence $\mu(0) = 0$ implies

$\mu(x) = 0$ for $0 \leq x \leq a$ which implies $\mu'(x) = 0$ for $0 \leq x \leq a$, and this is a contradiction since from (7.4) $\mu'(a) = -1$.

We can now conclude that the Volterra integral equation (7.8) is invertible. Hence if we solve (7.8) for $g(t)$ and substitute this back into (7.6), we have the (unique) solution to (7.7):

Theorem 7.1: There exists a unique solution of the initial-boundary value problem (7.7).

Proof: [10], [21].

We note in passing that Theorem 7.1 can easily be extended to treat the case of pseudoparabolic equations in one space variable with variable coefficients ([10],[21]). The concept of a Riemann function for pseudoparabolic equations can furthermore be extended to the case of pseudoparabolic equations in two space variables ([10], [22], [45]).

We now turn our attention to the case when (7.1) is defined in a cylindrical domain $D \times (0,T)$ where D $\mathbb{R}^n$, $n \geq 2$. For simplicity we again assume that all the constants appearing in (7.1) are equal to unity and set $n = 3$. (Other values of n can be handled with slight modifications). We assume that D is a bounded, simply connected domain with smooth boundary ∂D, and consider the initial-boundary value problem

$$\begin{aligned} &\Delta_3 u_t - u_t + \Delta_3 u = 0 \quad \text{in} \quad D \times (0,T) \\ &u(\underset{\sim}{x},t) = f(\underset{\sim}{x},t) \quad \text{on} \quad \partial D \times [0,T] \\ &u(\underset{\sim}{x},0) = 0 \quad \text{in} \quad D \end{aligned} \tag{7.9}$$

where $f(\underset{\sim}{x},t)$ is a continuously differentiable function prescribed on $\partial D \times [0,T]$. We note that the case of non-homogeneous initial data can be reduced to a problem of the form (7.9) by first solving an initial value problem by means of the Fourier transform (c.f.[62]). From the maximum principle for pseudoparabolic equations ([71], [72]) we have that a solution to (7.9), if it exists, is unique, and hence our problem is to derive a constructive method for obtaining the solution to (7.9). We shall do this by the use of a fundamental solution and the method of integral equations (we note that this approach can also be used for pseudoparabolic equations with variable coefficients ([64]). We define a fundamental solution of $\Delta_3 u_t - u_t + \Delta_3 u = 0$ by the formula ([23])

$$\begin{aligned} \Gamma(\underset{\sim}{x},t;\underset{\sim}{\xi},\tau) &= -\frac{1}{\pi i} \oint_{|\mu-1|=\varepsilon} \frac{1}{R} \exp\left[-\mu R + \frac{\mu^2}{1-\mu^2}(t-\tau)\right] d\mu \\ &= \frac{e^{-R}}{R} \sum_{n=1}^{\infty} a_n(\underset{\sim}{x};\underset{\sim}{\xi})(t-\tau)^n \end{aligned} \tag{7.10}$$

where $R = |\underset{\sim}{x}-\underset{\sim}{\xi}|$ and $a_1(\underset{\sim}{x};\xi) = 1$, and look for a solution of (7.9) in the form

$$u(\underset{\sim}{x},t) = \frac{1}{2\pi} \int_0^t \int_{\partial D} \rho(\underset{\sim}{\xi},\tau) \frac{\partial^2 \Gamma}{\partial \nu \partial \tau} (\underset{\sim}{x},t;\underset{\sim}{\xi},\tau) ds d\tau \tag{7.11}$$

where ν is the unit normal pointing into D and $\rho(\underset{\sim}{\xi},\tau)$ is a continuous density to be determined. Then from the known discontinuity properties of double layer potentials for metaharmonic functions we have from (7.11) that

$$
\begin{aligned}
f_t(\underset{\sim}{x},t) = \rho(\underset{\sim}{x},t) &- \int_{\partial D} \rho(\underset{\sim}{\xi},t)\frac{\partial}{\partial\nu}\frac{e^{-R}}{R}\,ds \\
&+ \int_0^t \rho(\underset{\sim}{x},\tau)\left[\sum_{n=2}^{\infty}\gamma_n(t-\tau)^{n-2}\right]d\tau \\
&+ \int_0^t \int_{\partial D} \rho(\underset{\sim}{\xi},\tau)\,\frac{\partial^3\Gamma}{\partial\nu\partial\tau\partial t}\,(\underset{\sim}{x},t;\underset{\sim}{\xi},\tau)\,ds\,d\tau
\end{aligned}
\tag{7.12}
$$

where the γ_n are known constants (c.f. [23]). The integral equation (7.12) is of the form (in the Banach space $C^0(\partial D \times [0,T])$)

$$
f_t = (\underset{\sim}{I}-\underset{\sim}{T}+\underset{\sim}{L}_1+\underset{\sim}{L}_2)\rho \tag{7.13}
$$

where $(\underset{\sim}{I}-\underset{\sim}{T})^{-1}$ exists (from the theory of metaharmonic potentials) and $\underset{\sim}{L}_1$ and $\underset{\sim}{L}_2$ are Volterra operators. Hence in $C^0(\partial D \times [0,T])$ $(\underset{\sim}{I}-\underset{\sim}{T}+\underset{\sim}{L}_1+\underset{\sim}{L}_2)^{-1}$ exists and $\rho = (\underset{\sim}{I}-\underset{\sim}{T}+\underset{\sim}{L}_1+\underset{\sim}{L}_2)^{-1}f_t$.

<u>Open Problem</u>: Consider oblique derivative problems and problems defined in domains with moving boundaries for pseudoparabolic equations.

<u>Open Problem</u>: Consider problems without initial conditions for pseudoparabolic equations (c.f. [73] for the case of the heat equation.)

VIII. Inverse Problems in the Theory of Heat Conduction.

In this section we shall consider two of the classical inverse problems for the heat equation, the problem of solving the heat equation backwards in time (or the final value problem for the heat equation) and the inverse Stefan problem. We shall restrict ourselves to the case of the heat equation in two space variables. Other inverse problems for the heat equation are also common, and the reader is referred to [2] and the references at the end of [61] for further information. We shall first consider the case of the final value problem for the heat equation and briefly indicate how an approximate solution can be obtained through the nethod of quasireversibility ([54], [58]) and the theory of pseudoparabolic equations as outlined in Section VII. The final value problem for the heat equation can be formulated as the problem of determining the temperature $u(x,y,t)$ in $D \times (0,1)$, where D is a bounded simply connected domain with smooth boundary ∂D, from the equations

$$u_{xx}+u_{yy} = u_t \quad \text{in} \quad D \times (0,1) \tag{8.1a}$$

$$u(x,y,t) = 0 \quad \text{on} \quad \partial D \times (0,1) \tag{8.1b}$$

$$u(x,y,1) = f(x,y) \tag{8.1c}$$

where $f(x,y)$ is a prescribed function vanishing on ∂D. In physical terms we are asking how must the body D be heated in order to have a prescribed temperature $f(x,y)$ at time $t = 1$. This problem is improperly posed in the sense that a solution does not in general exist, even for $f(x,y)$ analytic, and if a solution does exist, it does not depend continuously on the "initial" data $f(x,y)$ (c.f. [61], [69]). However if instead of (8.1a) we consider the pseudoparabolic equation

$$\beta\Delta_2 u_t+\Delta_2 u-u_t = 0 \tag{8.2}$$

where β is a small positive constant, we have from Section VII that (8.2), (8.1b), (8.1c) is a properly posed problem and can be solved by the method of integral equations (if the Fourier transform is first used to transform (8.2), (8.1b), (8.1c) to a non-homogeneous boundary value problem with homogeneous initial conditions). The question, of course, is what relation does the solution of (8.2), (8.1b), (8.1c) have with the solution (if it exists!) of (8.1a), (8.1b), (8.1c)? The answer to this question is contained in the following theorem of Ewing:

<u>Theorem 8.1</u>: Let $u(x,y,t)$ be a solution of (8.1a), (8.1b) such that $||u(x,y,1) - f(x,y)||_{L^2} < \varepsilon$, $||u(x,y,0)||_{L^2} < M$, where ε, M are positive constants, and let $v(x,y,t)$ be a solution of (8.2), (8.1b), (8.1c). Choose $\beta = 1/\log(M/\varepsilon)$. Then for every $t > 0$,

$$||u-v||_{L^2} = O([-\log(\varepsilon/M)]^{-1})$$

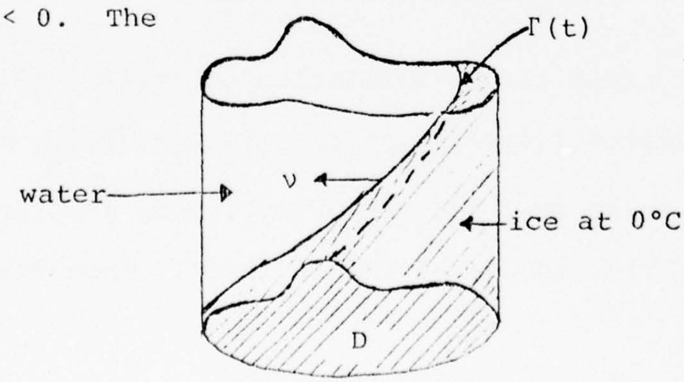
where the constant implicit in the "O" notation depends on t.

<u>Proof</u>: [37].

Note that the constant M in Theorem 8.1 can be determined in practice from a rough estimate of the maximum temperature attainable in the body D during the time interval [0,1].

For other methods for solving the heat equation backwards in time see [61] and [7].

We now turn our attention to the inverse Stefan problem for the heat equation in two space variables ([11], [20]). Assume that a bounded simply connected domain D with boundary ∂D is filled with ice at 0°C. Beginning at time $t = 0$ a non-negative temperature $\gamma = \gamma(x,t,t)$ (where $\gamma(x,y,0) = 0$) is applied to ∂D. The ice begins to melt and we shall let the interphase boundary $\Gamma(t)$ between ice and water be described by $\Gamma(t) = \{(x,y) : \Phi(x,y,t) = 0\}$, with the water lying in the region $\Phi(x,y,t) < 0$. The



(normalized) equations determining the temperature $u(x,y,t)$ of the water are

$$u_{xx}+u_{yy} = u_t \quad \text{in} \quad \Phi(x,y,t) < 0 \tag{8.3a}$$

$$u = \gamma \quad \text{on} \quad \partial D \times [0,T] \tag{8.3b}$$

$$u\Big|_{\Gamma(t)} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\Gamma(t)} = \frac{1}{|\nabla\Phi|}\frac{\partial \Phi}{\partial t}\Big|_{\Gamma(t)} \tag{8.3c}$$

where ν is the normal, with respect to the space variables, that points into the region $\Phi(x,y,t) < 0$, and the gradient is taken with respect to the space variables. The inverse Stefan problem is to find $\gamma(x,y,t)$ given $\Phi(x,y,t)$, i.e. how must D be heated so that the ice melts along a prescribed path? This problem is also improperly posed in the sense that in general a solution does not exist, even if $\Phi(x,y,t)$ is analytic. In the following we want to obtain sufficient conditions on $\Phi(x,y,t)$ such that a solution does exist to the inverse Stefan problem. Our approach also yields a constructive method for obtaining $\gamma(x,y,t)$ ([11], [20]) but we shall not pursue this here.

For each fixed t, $t \in [0,T]$, let $z = \phi(\zeta,t)$ map the unit disc conformally onto a domain D_t such that $D_t \supset D$ and define $\overline{\phi}(\zeta,t) = \overline{\phi(\overline{\zeta},t)}$ for t real. We shall show that a solution to the inverse Stefan problem exists if

$$\begin{aligned}\Phi(x,y,t) &= \operatorname{Im} \phi^{-1}(z,t) \\ &= \frac{1}{2i}\,[\phi^{-1}(z,t)-\overline{\phi}^{-1}(\overline{z},t)].\end{aligned} \tag{8.4}$$

It is assumed that $\phi(\delta,t)$ depends analytically on the parameter t.

Note that under the assumption (8.4) a <u>local</u> solution to (8.3a), (8.3c) exists by the Cauchy-Kowalewski Theorem. Our problem is to establish the existence of a <u>global</u> solution, i.e. to analytically continue the solution of a non-characteristic Cauchy problem for the heat equation. Let $z = x+iy$, $z^* = x-iy$ (note that $z^* = \bar{z}$ if and only if x and y are real) and define

$$U(z,z^*,t) = u\left[\frac{z+z^*}{2}, \frac{z-z^*}{2i}, t\right]$$

$$V(z,z^*,t) = \frac{1}{t-\tau} \exp\left\{\frac{(z-\xi)(z^*-\bar{\xi})}{4(t-\tau)}\right\} . \tag{8.5}$$

Then $U(z,z^*,t)$ and $V(z,z^*,t)$ are solutions of

$$L[U] \equiv U_{zz^*} - \frac{1}{4} U_t = 0$$

$$M[V] \equiv V_{zz^*} + \frac{1}{4} V_t = 0 \tag{8.6}$$

respectively for $t \neq \tau$. Let τ be real and for t on the circle $|t-\tau| = \delta$, $\delta > 0$, let $G(t)$ be a cell whose boundary consists of a curve $C(t)$ lying on the surface $\phi^{-1}(z,t) = \bar{\phi}^{-1}(z^*,t)$ and line segments lying on the characteristic planes $z = \xi$ and $z^* = \bar{\xi}$ respectively which join the point $(\xi,\bar{\xi})$ to $C(t)$. Integrating $VL[U]-UM[V]$ over the torus $\{(z,z^*,t):(z,z^*) \in G(t), |t-\tau| = \delta\}$ gives

$$U(\xi,\bar{\xi},\tau) = \frac{1}{4\pi i} \int_{|t-\tau|=\delta} \int_{C(t)} [VU_z dz - VU_{z^*} dz^*] dt$$

$$= \sum_{n=0}^{\infty} \text{residues} \tag{8.7}$$

where U_z and U_{z^*} can be computed from the Cauchy data (8.3c) (c.f. [11], [20]). Due to (8.4), it can be shown ([11], [20]) that $U(\xi,\bar{\xi},\tau)$ as defined by (8.7) is analytic in a region containing $D \times [0,T]$ and (8.7) gives the solution of the inverse Stefan problem if we evaluate it on $\partial D \times [0,T]$. Note that in order to get a physically meaningful solution of the inverse Stefan problem we assume that $\gamma(x,y,t) = 0$ on $\partial D \times [0,T] \cap \Phi \geq 0$ and choose the conformal mappings $\phi(\zeta,t)$ such that $u(x,y,t) \geq 0$ for $\Phi < 0$.

For a partial extension of these results to parabolic equations in three space variables see [70].

Open Problem: Can the inverse Stefan problem be approximated by appropriate solutions of pseudoparabolic equations?

Open Problem: What are *necessary* conditions for $\Gamma(t)$ to be an interphase boundary for the Stefan problem? For a conjecture in the case of one space variable see [48].

IX. Runge's Theorem for Parabolic Equations in Two Space Variables.

In this last section we shall outline how the method of integral operators can be used to obtain a version of Runge's Theorem for parabolic equations in two space variables. Our approach is a generalization of that used by Bergman ([3]) and Vekua ([75]) to derive a Runge Theorem for elliptic equations in two independent variables, and the new problems created in the present case are that classical solutions of parabolic equations with analytic coefficients are not necessarily analytic in the time variable, and in the case of analytic solutions we are forced to work with analytic functions of two complex variables instead of one complex variable as in the case of elliptic equations. A further problem is the need to develop an analogue for parabolic equations of the Riemann function for hyperbolic equations. Details of the results in this section can be found in [11] and [19]. For Runge's Theorem for parabolic equations in one space variable see Theorem 2.1 and Theorem 3.3 of these lectures.

We consider the parabolic equation

$$u_{xx}+u_{yy}+a(x,y,t)u_x+b(x,y,t)u_y+c(x,y,t)u = d(x,y,t)u_t \tag{9.1}$$

defined in a cylinder $D \times (0,T)$ where D is bounded simply connected domain (we assume without loss of generality that D contains the origin) and we make the assumption that the coefficients of (9.1) are real valued for x,y and t real and are entire funcitons of their independent complex variables. We also assume that $d(x,y,t) > 0$ in $D \times (0,T)$. Let $\bar{D}_0 \subset D$ be simply connected and compact, $\delta > 0$, and $D \supset D_1 \supset \bar{D}_0$ where D_1 is simply connected and such that ∂D_1 is analytic.

Theorem 9.1: For every $\varepsilon > 0$ there exists an analytic solution $u_1(x,y,t)$ of (9.1) in $D_1 \times (0,T)$ such that

$$\max_{\bar{D}_1 \times [\delta/2, T-\delta/2]} |u_1 - u| < \varepsilon .$$

Proof ([11], [19]): From the Weierstrass approximation theorem and the maximum principle we can approximate $u(x,y,t)$ by a solution $u_1(x,y,t)$ assuming analytic boundary data on ∂D. The Theorem now follows from the fact that solutions of (9.1) assuming analytic Dirichlet data on an analytic boundary are analytic (c.f. [39]).

Now make the change of variables

$$\begin{aligned} z &= x+iy \\ z^* &= x-iy \end{aligned} \tag{9.2}$$

mapping $\mathbb{C}^2$, the space of two complex variables, onto itself. Then (9.1) becomes

$$L[U] \equiv U_{zz^*} + A(z,z^*,t)U_z + B(z,z^*,t)U_{z^*} + C(z,z^*,t)U - D(z,z^*,t)U_z = 0 \tag{9.3}$$

where $U(z,z^*,t) = u(x,y,t)$, $A = \frac{1}{4}(a+ib)$, $B = \frac{1}{4}(a-ib)$, $C = \frac{1}{4}c$, and $D = \frac{1}{4}d$.

Theorem 9.2: Let $u_1(x,y,t) = U_1(z,z^*,t)$ be an analytic solution of (9.1) in a neighborhood of the origin. Then there exists a kernel $E(z,z^*,t,\tau,s)$ (which is independent of $u_1(x,y,t)$) and an analytic function $f(z,t)$ such that

$$u_1(x,y,t) = \mathrm{Re}\ \underset{\sim}{P}\{f\}$$

$$= \mathrm{Re}\left[-\frac{1}{2\pi i}\exp\left\{-\int_0^{\overline{z}} A(z,\sigma,t)\,d\sigma\right\}\cdot\right.$$

$$\left.\cdot\oint_{|t-\tau|=\delta}\int_{-1}^{1} E(z,\overline{z},t,\tau,s)\,f(z/2(1-s^2),\tau)\frac{ds\,d\tau}{\sqrt{1-s^2}}\right].$$

Furthermore, $f(\frac{z}{2}, t)$ and $U_1(z,0,t)$ have the same domain of regularity. $E(z,z^*,t,\tau,s)$ is an entire function of its independent complex variables except for an essential singularity at $t = \tau$.

<u>Proof ([11], [18], [19])</u>: Substitute $P\{f\}$ into $L[U] = 0$ and integrate by parts to obtain a differential equation and initial conditions satisfied by $E(z,z^*,t,\tau,s)$. This initial value problem can be solved by iteration.

By using the operator $\underset{\sim}{P}$ *we can construct the* <u>complex</u> <u>Riemann function</u> for $L[U] = 0$ ([11], [19], [49]) which is defined as the solution of the singular initial value problem

$$M[R] \equiv R_{zz^*} - \frac{\partial(AR)}{\partial z} - \frac{\partial(BR)}{\partial z^*} + CR + \frac{\partial}{\partial t}(DR) = 0$$

$$R(z,z^*,t)\Big|_{z^*=\xi^*} = \frac{1}{t-\tau}\exp\left\{\int_{\xi}^{z} B(\sigma,\xi^*,t)\,d\sigma\right\} \qquad (9.4)$$

$$R(z,z^*,t)\Big|_{z=\xi} = \frac{1}{t-\tau}\exp\left\{\int_{\xi^*}^{z^*} A(\xi,\sigma,t)\,d\sigma\right\}.$$

<u>Example</u>: For the heat equation we have (see Section VIII)

$$R(z,z^*,t) = \frac{1}{t-\tau} \exp\left\{\frac{(z-\xi)(z^*-\xi^*)}{4(t-\tau)}\right\} .$$

The complex Riemann function is needed in order to obtain the following theorem:

<u>Theorem 9.3</u>: Let $u_1(x,y,t)$ be a solution of (9.1) such that $u_1(x,y,t)$ is analytic for $(x,y) \varepsilon D_1$, $t \varepsilon \mathcal{E}$ where $\mathcal{E}$ is a neighborhood of $[\delta,T-\delta]$. Then $U_1(z,z^*,t) (= u_1(x,y,t))$ is analytic in $D_1 \times D_1^* \times \mathcal{E}$ where $D_1 = \{z : z \varepsilon D_1\}$, $D_1^* = \{z^* : \bar{z}^* \varepsilon D_1\}$.

<u>Proof ([11], [19])</u>: Use Stokes Theorem to represent $u_1(x,y,t)$ in terms of the complex Riemann function, where the domain of integration is an appropriate three dimensional torus situated in $\mathbb{C}^3$, the space of three complex variables. The regularity of $U_1(z,z^*,t)$ now follows from the regularity properties of the complex Riemann function, which is an entire function of its independent complex variables except for an essential singularity at $t = \tau$.

<u>Example</u>: $u_1(x,y) = (1+x^2+y^2)^{-1}$ (this is not a solution of (9.1) for any choice of the coefficients!) is real analytic for all x and y, in particular in $D_1 = \{(x,y): x^2+y^2<2\}$. However as a function of z and z^*, $U(z,z^*) = u(x,y)$ is singular in $D_1 \times D_1^*$, for example at $(i,i) \varepsilon D_1 \times D_1^*$.

<u>Theorem 9.4 (Runge's Theorem)</u>: Let $u(x,y,t)$ be a solution of (9.1) in $D \times (0,T)$. Then for every $\varepsilon > 0$ there exists an entire solution $u_2(x,y,t)$ of (9.1) such that for $\bar{D}_0 \subset D_1 \subset D$

$$\max_{\bar{D}_0 \times [\delta, T-\delta]} |u-u_2| < \varepsilon .$$

<u>Proof ([11], [19])</u>: Theorem 9.1 implies that $u(x,y,t)$ can be approximated on $\bar{D}_1 \times [\delta/2, T-\delta/2]$ by an analytic solution $u_1(x,y,t)$ of (9.1). Theorems 9.2 and 9.3 imply that $u_1(x,y,t) = \text{Re}\, \underset{\sim}{P}\{f\}$ where $f(z,t)$ is analytic in the product domain $D_1 \times \mathcal{E}$. Since product domains are Runge domains (c.f. [77]) $f(z,t)$ can be approximated by a polynomial $f_n(z,t)$ on $\bar{D}_0 \times \bar{\mathcal{E}}_0$ where $[\delta, T-\delta] \subset \bar{\mathcal{E}}_0 \subset \mathcal{E}$. Hence we can choose $u_2(x,y,t) = \text{Re}\, \underset{\sim}{P}\{f_n\}$ for n sufficiently large.

<u>Open Problem</u>: Derive Runge's Theorem for second order parabolic equations in $n \geq 3$ space variables (c.f. [32] and the references cited there).

<u>Open Problem</u>: In Theorem 9.4 can $\bar{D}_0 \times [\delta, T-\delta]$ be replaced by $\bar{D} \times [0,T]$? Can the cylindrical domain be replaced by a domain with moving boundary?

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